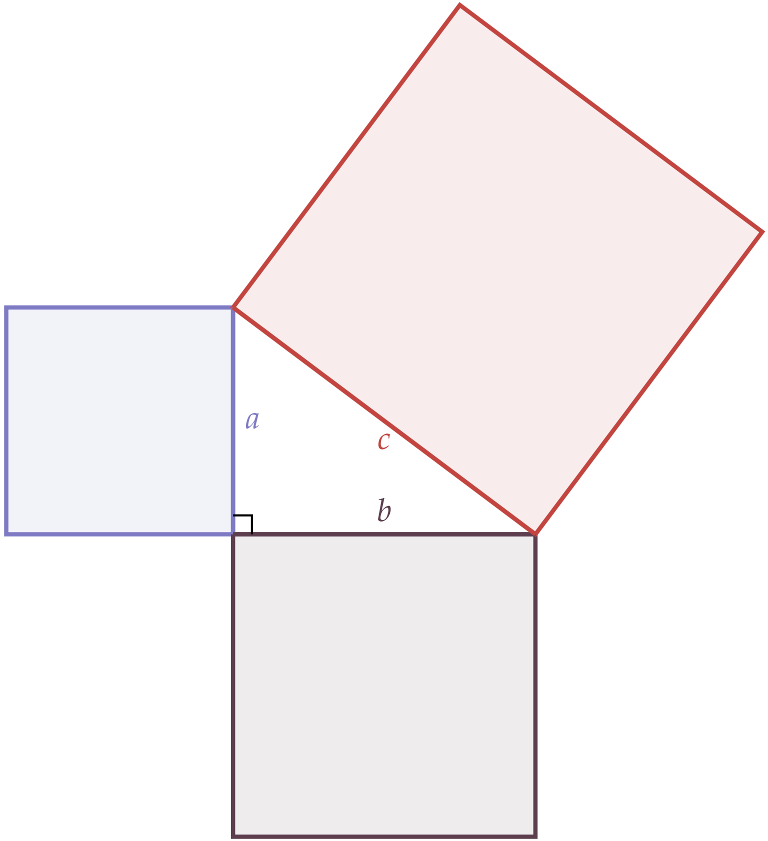
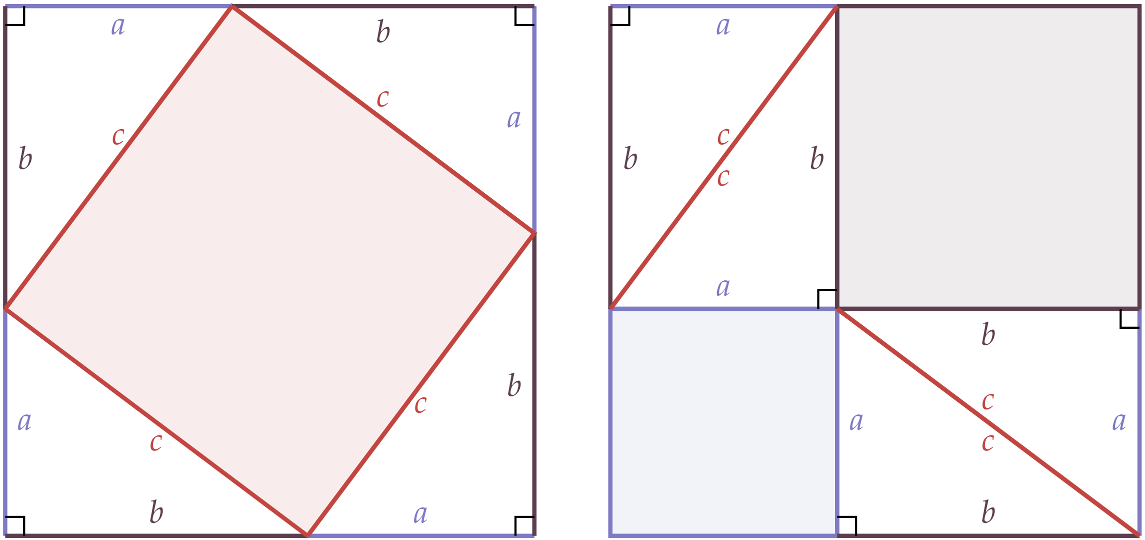
Geometric Proofs of the Pythagorean Theorem



1 Pythagorean proof

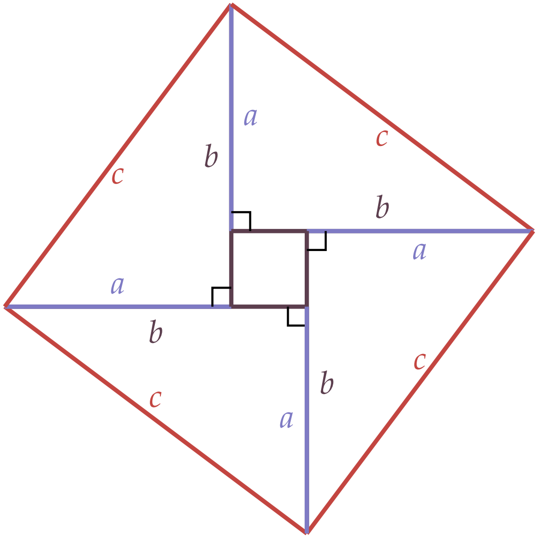


Arrange four similar right triangles in a square with side length $a + b$, so that the area not covered by the triangles is of area c^2. Rearrange the triangles so that the area uncovered is of area $a^2 + b^2$.

- Discovered independently by Bhāksara II.
- Also discovered in Zhōu dynasty China and recorded in the *Zhōubìsuàn Jīng*.

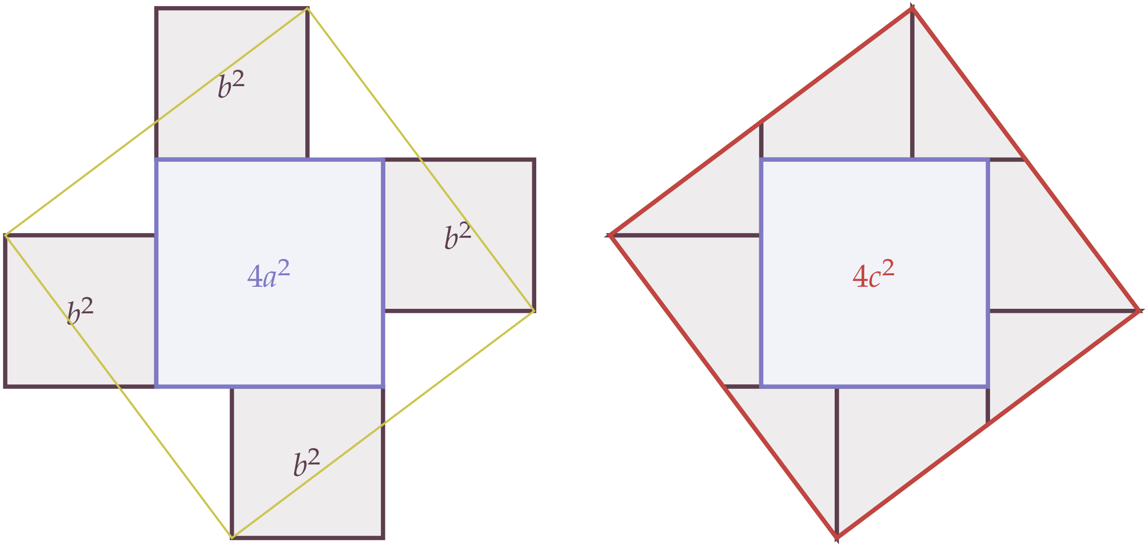
Alternatively, note that the area of the left square is $(a+b)^2$, and it is covered by tiles of area $4 \cdot \frac{1}{2}ab + c^2$. Thus, $a^2 + 2ab + b^2 = 2ab + c^2$, and $a^2 + b^2 = c^2$.

2 Pythagorean proof (variant)



The area of the square is c^2. It is covered by four triangles of area $\frac{1}{2}ab$, and a square of area $(b-a)^2 = b^2 - 2ab + a^2$. Thus, $c^2 = 2ab + b^2 - 2ab + a^2 = a^2 + b^2$.

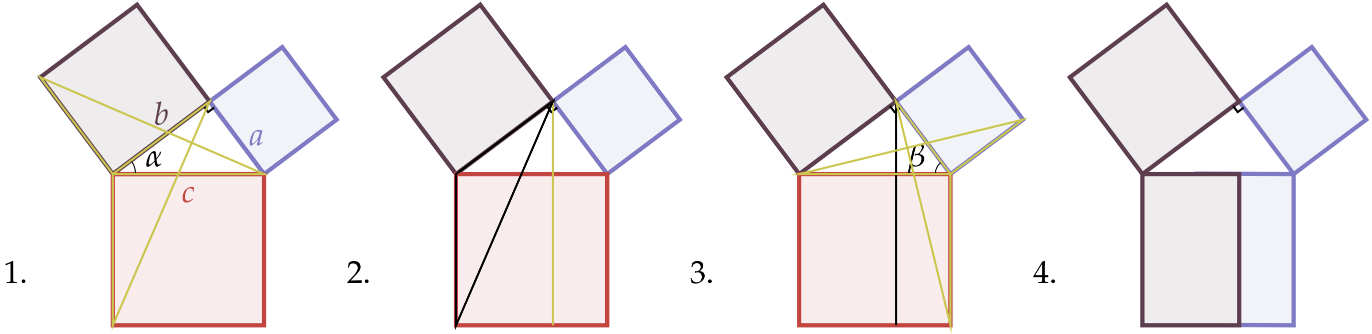
3 Alandete's proof



Arrange four squares of side length b around a square of side length $2a$. The length of each convex side is $(b+2a)-b=2a$. Draw a square at the corners of the former squares. This forms four triangles of side lengths a, b, and c, and four triangle-shaped gaps. Move the triangles into the gaps to form a square of side length $2c$. Thus, $(2a)^2 + 4 \cdot b^2 = (2c)^2$, and $a^2 + b^2 = c^2$.

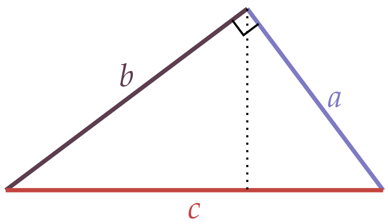
- Discovered by Edgardo Alandete.

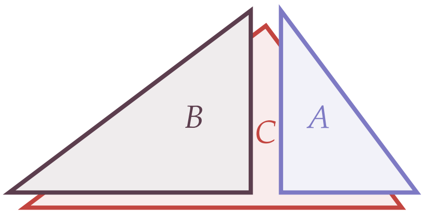
4 Euclid's proof



Draw two lines to form two congruent obtuse triangles with sides b and c and interior angle $\alpha + 90°$ (1). The area of this triangle is $\frac{1}{2}b^2$, since its base and height are b. Drop a vertical line from the right angle, dividing the square with area c^2 into left and right rectangles (2). The left rectangle has twice the area of the triangle, since the base of the triangle is c and it's height is the width of the rectangle. Thus, the left rectangle has area b^2. By the same reasoning, the right rectangle has twice the area of the triangle with area $\frac{1}{2}a^2$ (3). Thus, $c^2 = a^2 + b^2$ (4).

5 Proof by bisection

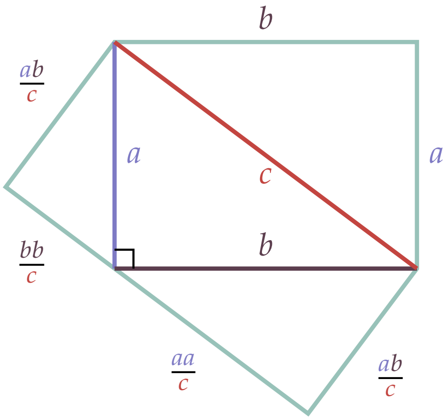




Bisect the triangle at its right angle perpendicular to the hypotenuse. The two right triangles formed are similar to the original. Thus, the squares of their hypotenuses are proportional to their areas, respectively. Since sum of the areas of triangles A and B equals the area of triangle C, the sum of the squares of their hypotenuses must also be equal: $a^2 + b^2 = c^2$.

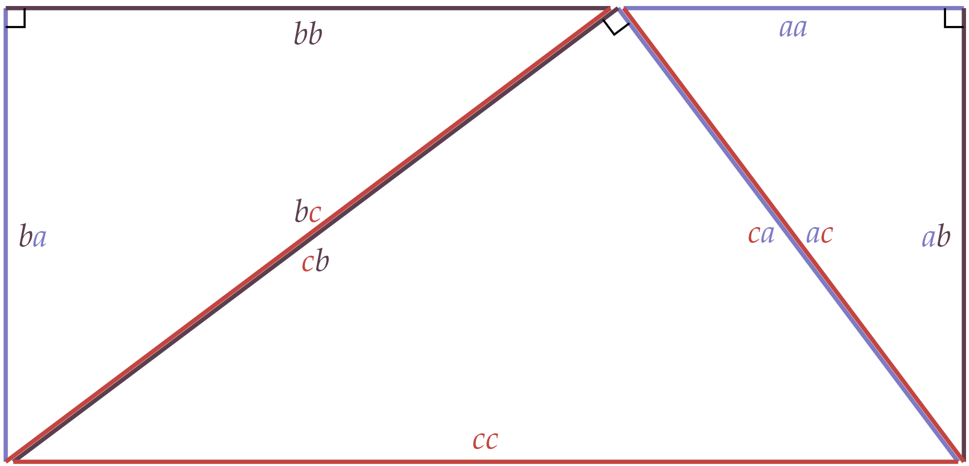
- Discovered independently by Einstein as a schoolboy.

6 Proof by rectangle construction



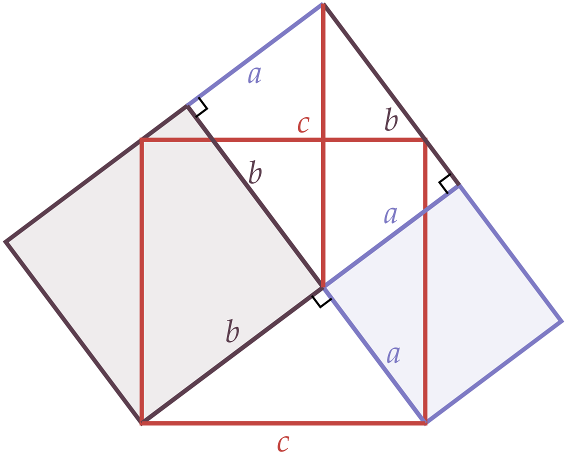
Draw a congruent triangle whose hypotenuse is c. Draw two triangles similar to this, whose hypotenuses are a and b. They form a rectangle whose long side is c, and $\frac{a^2}{c} + \frac{b^2}{c}$.

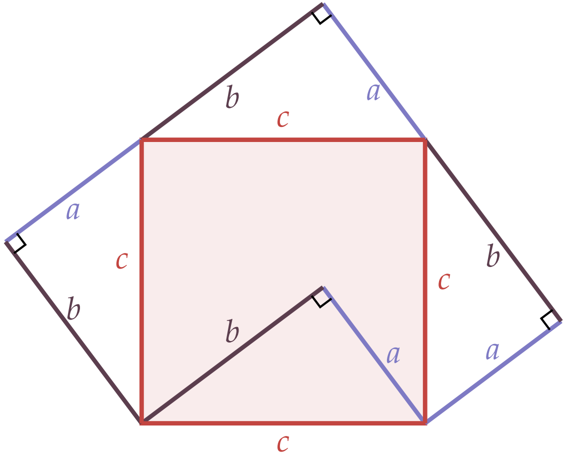
7 Proof by scaling



Take three right triangles and scale one by a, one by b, and one by c. They can be arranged into a rectangle so that sides ac and bc correspond. Then one side of the rectangle is of length $aa + bb$ and its opposite is of length cc. So $a^2 + b^2 = c^2$.

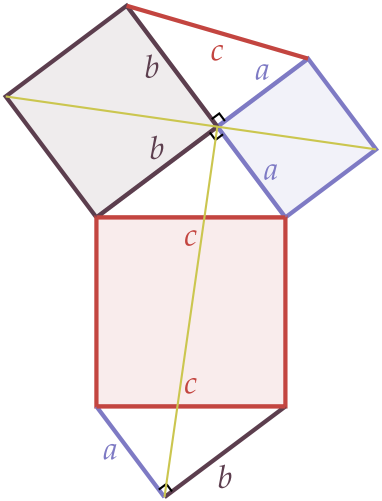
8 Thābit ibn Qurrah’s proof





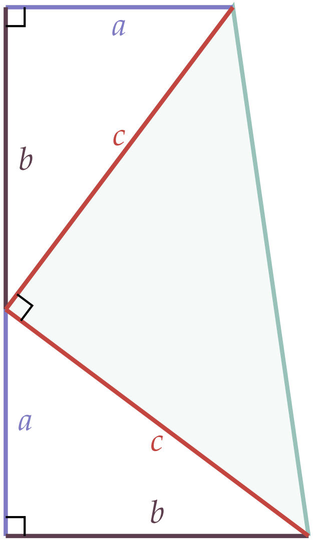
Draw a figure consisting of two squares on sides a and b, and between them two right triangles, each congruent to the original. The area of the figure minus the three triangles is $a^2 + b^2$. Draw a square on side c, inscribed in the figure. The parts of the figure outside this square are triangles congruent to the original, since they have the same angles and hypotenuse c. The area of the figure minus these three triangles is c^2.

9 Da Vinci's proof



Draw a square on the hypotenuse, and a similar right triangle on the opposite side of the square, forming a hexagon. Draw two squares on the legs, and a line between them, forming another hexagon with the original triangle. Divide each hexagon along the main diagonal into two quadrilaterals. Since each quadrilateral has side lengths b, c, a, and identical angles, they are all congruent. Thus, the two hexagons have equal area. The area of the first, minus the two triangles, is c^2. The area of the second, minus the two triangles, is $a^2 + b^2$.

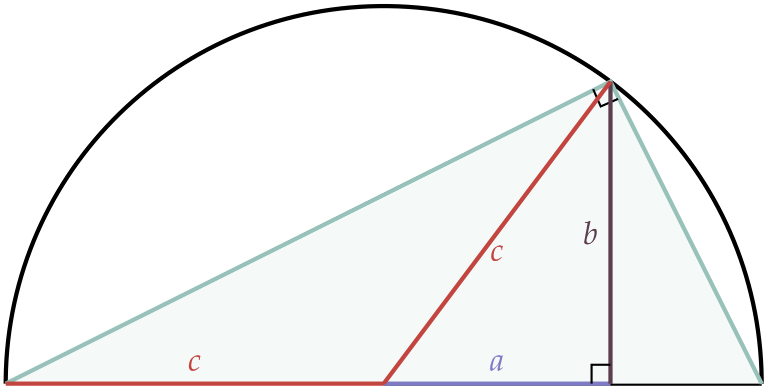
10 Garfield's proof



The area of the trapezoid is $(a+b)\cdot\frac{1}{2}(a+b)$, and the sum of the areas of the three triangles is $2(\frac{1}{2}ab)+\frac{1}{2}c^2$. Thus $\frac{1}{2}(a+b)^2 = ab + \frac{1}{2}c^2$; so $(a+b)^2 - 2ab = c^2$, and $a^2 + b^2 = c^2$.

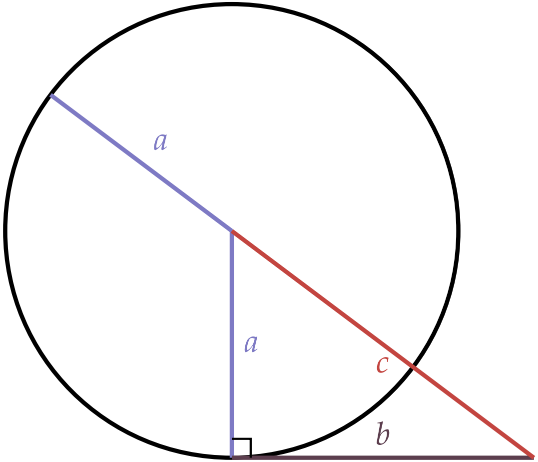
- Discovered by U.S. President James Garfield in 1876.

11 Proof by circumscription



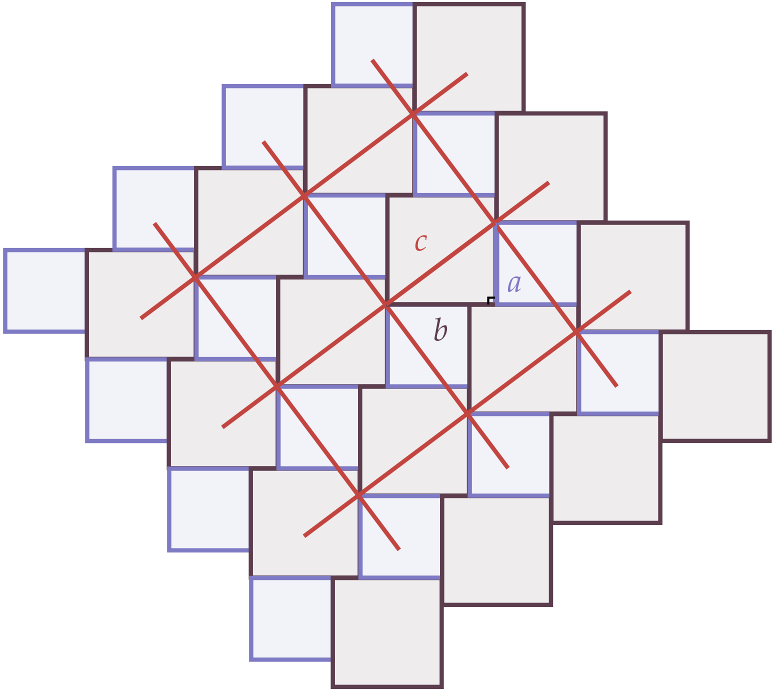
Inscribe the triangle in a circle of radius c. This defines another right triangle whose hypotenuse is the diameter of the circle, and whose height b splits the diameter into $c+a$ and $c-a$. Then by the intersecting chords theorem, $b \cdot b = (c+a)(c-a)$, and $b^2 = c^2 - a^2$.

12 Proof by tangency



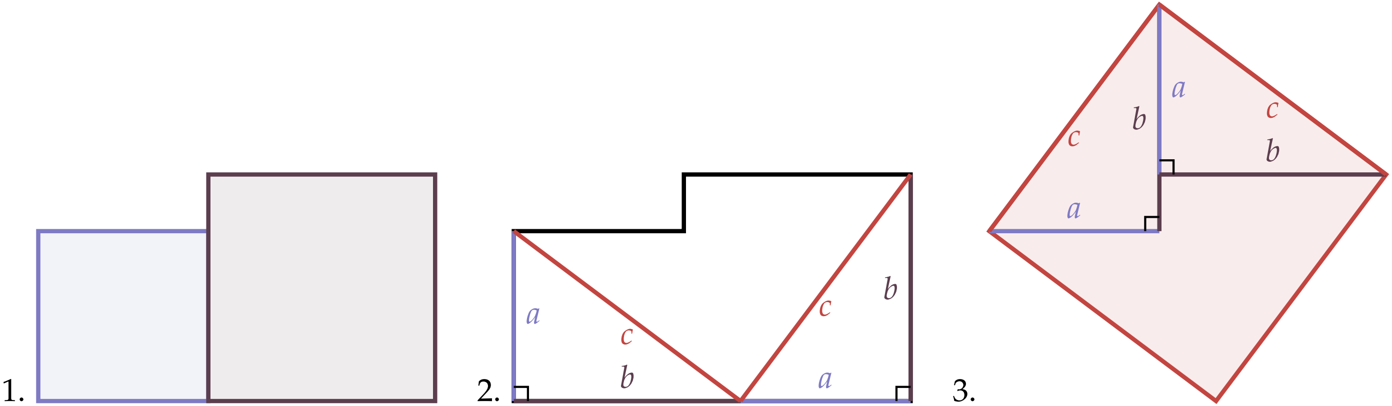
Draw a circle of radius a around a corner of the triangle, so that side b is tangent to it. Applying the power of a point theorem to the other non-right-angled corner, $(c + a)(c - a) = b \cdot b$, and $c^2 - a^2 = b^2$.

13 Proof by tesselation



Tessellate the plane with squares of side length a and b. Where these sides meet in the larger square, draw a hypotenuse c. Superimpose a grid of squares of side length c on the plane. Each grid cell of area c^2 contains one square of area a^2 and one of area b^2.

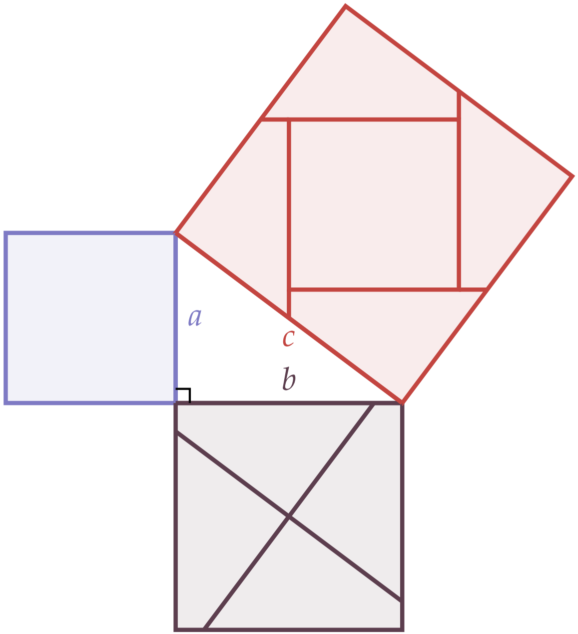
14 Proof by hinged dissection



Draw two squares of area a^2 and b^2 (1). Draw two triangles inside them along the edges (2). Rearrange the three pieces to form a square of area c^2 (3).

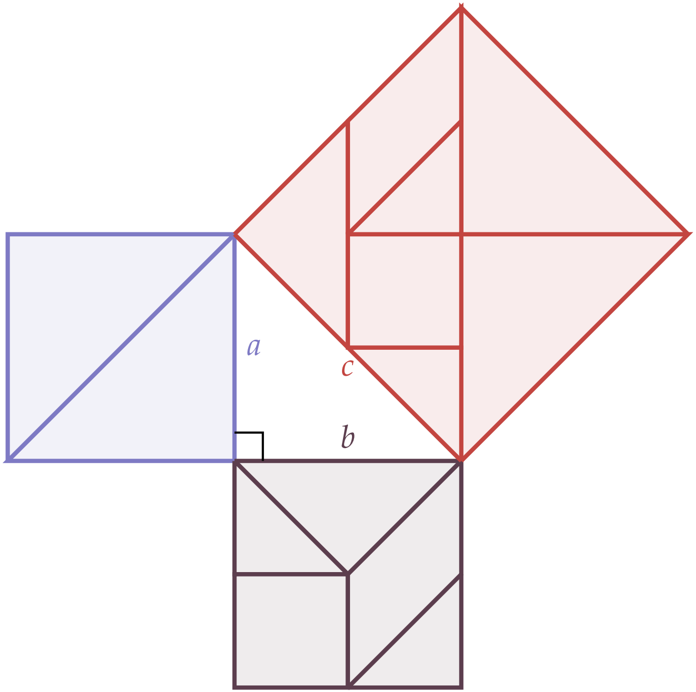
- Discovered by Thābit ibn Qurrah in the 9th century.

15 Perigal's proof

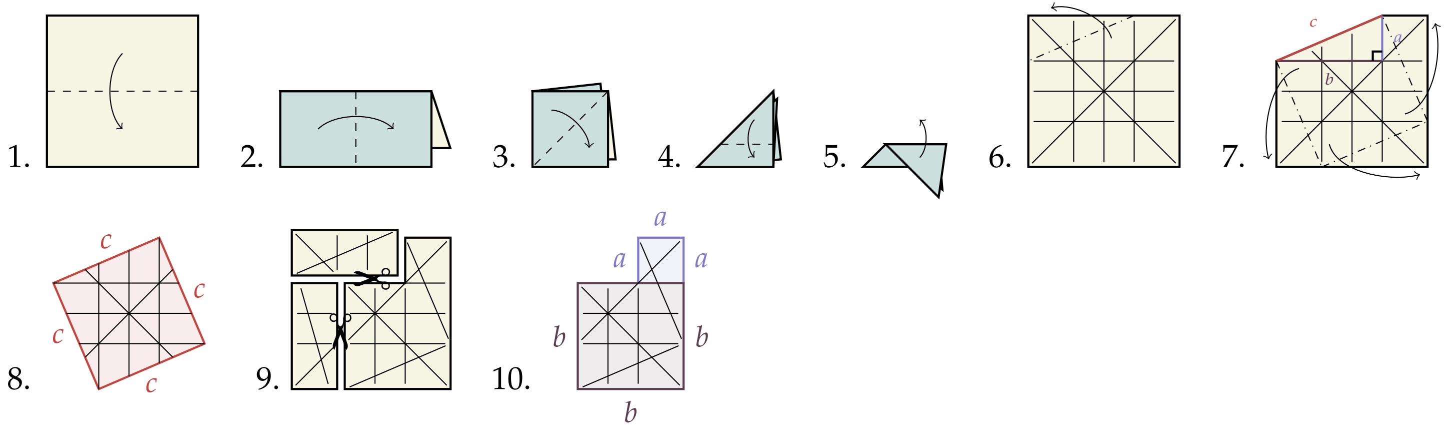


- Discovered in 1872 by Henry Perigal, and engraved on his tombstone in Essex.

16 Tangram proof



17 Origami proof



Fold origami paper in half twice (1–2), then along the diagonal (3). Fold the top corner down—the height of this crease determines the ratio of a to b (4). Unfold all (5). Fold the top corner back along the diagonal—this is the first triangle (6). Repeat with the other three corners (7). The remaining square has area c^2 (8). Unfold, and cut out four triangles from the top and left (9). The remaining paper has area $a^2 + b^2$ (10).

- Posted to YouTube by Vi Hart in 2011.